

THE PROJECTIVE SYMPLECTIC GEOMETRY OF HIGHER ORDER VARIATIONAL PROBLEMS: MINIMALITY CONDITIONS

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ABSTRACT. We associate curves of isotropic, Lagrangian and coisotropic subspaces to higher order, one parameter variational problems. Minimality and conjugacy properties of extremals are described in terms of these curves.

1. INTRODUCTION

We introduce the subject by briefly describing the classical case of Riemannian manifolds: Let (M, g) be Riemannian manifold, and $E = \frac{1}{2}g(v, v)$ its associated energy function. On the space $\Omega_{p,q}$ of smooth curves γ joining $p = \gamma(a)$ with $q = \gamma(b)$, we consider the variational problem with fixed endpoints associated to the action functional $A(\gamma) = \int_a^b E(\gamma'(s)) ds$. The local minimality of a geodesic γ is assured, using the second variation formula, by the strong positive definiteness of the Hessian of the functional, which in turn is given by the non-existence of conjugate points on the interval $(a, b]$.

It is well-known that the constructions mentioned in the previous paragraph can be studied via the local, global and self-intersection properties of certain curves of Lagrangian planes in a symplectic vector space. This viewpoint, in addition to be of great value in understanding purely Riemannian geometry [20, 25], is particularly suited to the study of more general variational problems, e.g., Finsler geometry [14], Lorentzian and semi-Riemannian geometry [19] and sub-Riemannian geometry [1].

The aim of this paper is to develop this projective symplectic viewpoint for *higher order* variational problems, that is, functionals associated to Lagrangians $L : J^k(\mathbb{R}, M) \rightarrow \mathbb{R}$ defined on the space of k -jets of curves on M , using a purely variational approach: directly from

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the Hessian of the given functional and its associated Euler-Lagrange equation, without detouring through non-holonomic mechanics. Higher order variational problems appear, for example in control theory (e.g. [12]) and applications to physics (e.g., higher-order mechanics considered in the book [18] and in the papers [6, 13]).

We use, extend and interpret in the symplectic-projective setting the line of classical identities of Cimino, Picone, Eastham, Eswaran, etc., [4, 9, 10, 15, 17, 24], which systematize the writing of higher order Lagrangians as a perfect square plus a total differential term. For the higher dimensional case, non-commutativity produces combinatorial difficulties in the establishment of the aforementioned identities. In this case we extend to arbitrary dimensions the work of Coppel [5], which uses the Legendre transform to produce a Hamiltonian version that is much more amenable to computation. As a by-product, we get a version of Eswaran's identity for the higher dimensional case.

We shall see that to such variational problems there is a curve of *isotropic* subspaces whose successive prolongations determine the conjugacy, positive definiteness of the Hessian and, therefore, the local minimality of the extremals.

As in the first order problem, the main actor for determining minimality and conjugacy is the self-intersection properties of a (prolonged) curve of *Lagrangian* subspaces which we call the *Jacobi curve*. However, in contrast with the first-order case, this curve is *always* degenerate, even for positive definite Lagrangians. Another purpose of this work is to precisely state the degree of degeneracy of the Jacobi curve for higher-order problems. The present paper focuses on projectivization of the sufficient conditions for minimality; more precise information concerning the index of the Hessian (that in particular furnishes necessary conditions) will be studied in the sequel [8].

The paper is organized as follows: a brief review of the preliminary setting up is done in section 2. The core of the paper is then given in sections 3 and 4, where the Jacobi curve is constructed and studied respectively for one-dimensional and n -dimensional, higher order variational problems. The reason for the separation is that the one-dimensional case can be treated using the one-dimensional classical identities, which makes it a good starting point to understand the structure of the Jacobi curve, whereas in the high dimensional case we shall use the Legendre transform in order to apply the work of Coppel [5], which we extend to the n -dimensional case.

2. PRELIMINARIES

2.1. The Hessian of a higher order variational problem. Let M^n be an n -dimensional manifold, and $L : J^k(\mathbb{R}, M) \rightarrow \mathbb{R}$ be a k -th order Lagrangian. We shall assume the *strong Legendre condition*, which is that the restriction of the Lagrangian L to the inverse image of the projection $J^k(\mathbb{R}, M) \rightarrow J^{k-1}(\mathbb{R}, M)$ is strictly convex. Such a Lagrangian takes the local form $L(q_0^A, \dots, q_k^A)$, where the subindex is the order of differentiation and the superindex the n -coordinates on M gives local coordinates on $J^k(\mathbb{R}, M)$, and the strong Legendre condition takes the form “ $\frac{\partial L}{\partial q_k^A \partial q_k^B}$ is positive definite”. From this data, we associate the functional

$$\mathcal{L}[\sigma] = \int_a^b L(j^k(\sigma)(s)) ds,$$

where $j^k\sigma(s)$ denotes the k -jet prolongation of σ ; in the coordinates given this is just the curve $(\sigma(s), \dot{\sigma}(s), \dots, \sigma^{(k)}(s))$. Take $p, q \in J^{k-1}(\mathbb{R}, M)$, and let us, as usual, restrict \mathcal{L} to the set of curves $\Omega_{p,q}[a, b]$ given by

$$\Omega_{p,q}[a, b] = \{\sigma : [a, b] \rightarrow M \mid j^{k-1}\sigma(a) = p \text{ and } j^{k-1}\sigma(b) = q\}.$$

Consider now an extremal $\gamma(t)$ of the functional \mathcal{L} . This curve satisfies the Euler-Lagrange equation, which might be understood in local coordinates (giving an elementary approach as in [11], 2.11), a symplectic approach as in [18, 26] or a more abstract one as in [3]. The approach taken makes no difference in what follows. For the standard infinite dimensional manifold theory needed to talk about differentiability we refer the reader to [16, 23].

The crucial point is that, if γ is a critical point of \mathcal{L} , that is, $d\mathcal{L}_\gamma = 0$ for some reasonable notion of differentiability, then the Hessian of \mathcal{L} at γ is intrinsically defined, for example as in [21, 22]. This Hessian is a quadratic form on the space $T_\gamma\Omega_{p,q}[a, b]$ of vector fields along $\gamma(t)$ which vanish up to order $k-1$ at the endpoints. Since we know *a priori* that the Hessian is well defined, we can compute in a convenient way: let us choose a frame $v_1(t), \dots, v_n(t)$ of vector fields along $\gamma(t)$ that span $T_{\gamma(t)}M$ at each point. In terms of this frame, each element $X(t) \in T_{\gamma(t)}M$ can be written as $X(t) = \sum_i h_i(t)v_i(t)$, and then $T_\gamma\Omega_{p,q}[a, b]$ can be identified (with respect to the frame v_1, \dots, v_n) with the space

$$C_k^\infty([a, b], \mathbb{R}^n) = \{h \in C^\infty([a, b], \mathbb{R}^n) \mid h^{(s)}(a) = h^{(s)}(b) = 0, \text{ for } 0 \leq s \leq k-1\}$$

then we can work in local coordinates do show that the Hessian of \mathcal{L} at an extremum is given by a quadratic form as follows:

$$\mathcal{Q}[h] = \int_a^b \sum_{1 \leq i \leq j \leq k} h^{(j)}(t)^\top Q_{ij}(t) h^{(i)}(t),$$

where each $Q_{ij}(t)$ is a smooth $n \times n$ -matrix valued function. The matrices Q_{ij} depend on the Lagrangian and its derivatives, but the only important information is that the top level $Q_{kk}(t)$ is symmetric and positive definite. These two conditions are independent of the chosen frame; if the frame comes from prolonged local coordinates, the term $Q_{kk}(t)$ is essentially the Hessian of the Lagrangian restricted to the coordinates q_k^A .

The aim of this paper is, starting from the classical identities of higher order calculus of variations, to *construct and study the projective curves whose lack of self-intersection imply that \mathcal{Q} is positive definite*. Let us remark that the absence of self-intersection of our Jacobi curves on $(a, b]$ will imply positive definiteness, but a simple step similar to sections 28 and 29.4 of [11] allows the passage from positive definite to *strongly* positive definite (that is, $\mathcal{Q}(h) \geq C|h|$ for some constant $C > 0$ and an appropriate C^k -norm for h). Thus the projective topology of the Jacobi curves (plus the approximation up to order 2 of the functional \mathcal{L}) does indeed provide sufficient conditions for local minimality.

2.2. Curves in the half - and divisible - Grassmannians and their rank. As mentioned in the introduction, Jacobi curves in the context of higher-order variational problems will always be degenerate. In this section we will furnish the tools to quantify this degeneracy.

The local geometry (as opposed to the global topology) of curves in Grassmann manifolds can be studied in terms of frames spanning the given curve. This is the approach used in [2] and [7], where the reader can find details about the construction of local invariants.

In this section we use this viewpoint to *rank* of a curve in the Grassmannian, which measures its degeneracy.

Recall that if V is a real vector space, the tangent space of a Grassmann manifold $Gr(k, V)$ can be canonically identified with the quotient vector space $T_\ell Gr(k, V) \cong \text{Hom}(\ell, V/\ell)$.

Definition 2.1. Let I be an interval and $\ell : I \rightarrow Gr(k, V)$ be a curve in a Grassmann manifold. The *rank* of ℓ at $t \in I$ is the rank of $\ell'(t)$ considered as an element of $\text{Hom}(\ell(t), V/\ell(t))$.

We now fix a basis on V so that the $V \cong \mathbb{R}^n$ and denote the Grassmannian by $Gr(k, n)$. Given a curve $\ell(t) \in Gr(k, n)$, we can lift it

to a curve of k linearly independent vectors $a_1(t), \dots, a_k(t)$ in \mathbb{R}^n that span $\ell(t)$. This we can codify as a $n \times k$ matrix $\mathcal{A}(t)$ whose columns are the vectors $a_i(t)$, $\mathcal{A}(t) = (a_1(t) | \dots | a_k(t))$ (the vertical bars denote juxtaposition) We have

Proposition 2.2. *In the situation above, the rank of $\ell(t)$ is given by $\text{rank}(\mathcal{A}(t) | \mathcal{A}'(t)) - k$.*

Proof. Let us recall concretely how the identification $T_\ell \text{Gr}(k, V) \cong \text{Hom}(\ell, V/\ell)$ is done, in a way that is useful for computations involving the frame once a basis of V is fixed: given a curve $\ell(t)$, choose a curve of idempotent matrices $\rho(t)$ representing projections (not necessarily orthogonal, since we do not assume any Euclidean structure in V) such that the image of $\rho(t)$ is $\ell(t)$. Then we have:

- (1) The derivative of a curve of projections $\rho(t)$ is a curve of endomorphisms that maps $\text{image}(\rho(t))$ into $\ker(\rho(t))$ and vice-versa.
- (2) The quotient $V/\ell(t)$ can be identified with $\ker \rho(t)$.

Then, the derivative $\rho'(t_0)$ provides a map from $\text{image}(\rho(t_0)) = \ell(t_0)$ into $\ker \rho(t_0) \cong V/\ell(t_0)$. It is straightforward, after unraveling the identifications, that this map is independent of the curve of projections chosen to represent $\ell(t)$.

Fix one such adapted curve of projection matrices $\rho(t)$. Then, since $\rho(t)\mathcal{A}(t) = \mathcal{A}(t)$, we have

$$\mathcal{A}'(t) = \rho'(t)\mathcal{A}(t) + \rho(t)\mathcal{A}'(t).$$

Juxtaposing and computing the rank, we get

$$\text{rank}(\mathcal{A}(t) | \mathcal{A}'(t)) = \text{rank}(\mathcal{A}(t) | \rho'(t)\mathcal{A}(t) + \rho(t)\mathcal{A}'(t)) = \text{rank}(\mathcal{A}(t) | \rho'(t)\mathcal{A}(t)),$$

where the last equality follows from $\text{image}(\rho(t)) = \ell(t)$ and therefore the columns $\rho(t)\mathcal{A}'(t)$ are in the span of the columns before the vertical bar. Now since the columns of $\mathcal{A}(t)$ are in the image of $\rho(t)$ and the columns $\rho'(t)\mathcal{A}(t)$ are in the kernel of $\rho(t)$, it follows that the rank of the right-hand side of the equation above is $k + \text{rank} \rho'(t)\mathcal{A}(t) = k + \text{rank} \ell(t)$. \square

In the case of the half-Grassmannian $\text{Gr}(k, 2k)$ a curve $\ell(t)$ is called *fanning* if it has maximal rank, that is the rank of $\ell(t)$ is k and $\ell'(t) : \ell(t) \rightarrow \mathbb{R}^{2k}/\ell(t)$ is invertible. This (generic) condition is the starting point of the invariants defined in [2]: the invariantly defined nilpotent endomorphism $\mathbf{F}(t)$ of \mathbb{R}^{2k} given by the composition $\mathbb{R}^{2k} \rightarrow \mathbb{R}^{2k}/\ell \xrightarrow{(\ell'(t))^{-1}} \ell(t) \hookrightarrow \mathbb{R}^{2k}$.

All of the invariants studied in [2] for the Grassmannian can be used for the local study of curves in the Lagrangian Grassmannian; there is a

single extra *discrete* invariant, the *signature*, that solves the equivalence problem for curves in the Lagrangian Grassmannian.

It is important to note that, in the case of higher order variational problems, we will construct a curve of Lagrangian subspaces whose self-intersections (or, rather, lack thereof) control the positivity of the second variation. However, in the context of higher order variational problems, this curve is *never* fanning in the sense of [2] thus a different approach to the the invariants must be taken. One possibility is using the theory developed by [28]. An alternative approach could be the use of curves $\ell(t)$ in a *divisible Grassmannian* $Gr(k, nk)$ and the invariant theory that has been developed in [7]. This is especially adapted to the projective geometry of higher order linear differential equations [27], such as the Euler-Lagrange equation of a quadratic Lagrangian of higher order. The concept of fanning curve in this case requires the $(n-1)$ -jet of the curve: if $\ell(t)$ is a curve in a divisible Grassmannian $Gr(k, nk)$ spanned as above by the columns of a $nk \times k$ matrix $\mathcal{A}(t)$, we say that ℓ is fanning if the matrix $(\mathcal{A}(t) | \mathcal{A}'(t) | \dots | \mathcal{A}^{(n-1)}(t))$ is invertible. This is, again, a generic condition on smooth curves in a divisible Grassmannian, and that is satisfied in our case of *isotropic* curves whose prolongation gives the curve in the Lagrangian Grassmannian.

An important construction for these curves is the *canonical linear flag* (compare [28])

$$Span\{A(t)\} \subset Span\{A(t), \dot{A}(t)\} \subset \dots,$$

which only depends on the curve $\ell(t)$ and its successive jets. We refer to each step of this sequence of inclusions as a *prolongation* of the curve ℓ .

Fanning curves in the divisible Grassmannian satisfy that the canonical linear flag jumps dimension by k at each stage up to the forced stabilization at maximal dimension nk .

3. THE SYMPLECTIC PROJECTIVE GEOMETRY OF HIGHER ORDER QUADRATIC FUNCTIONALS: THE ONE DIMENSIONAL CASE

Let $a < b \in \mathbb{R}$. Consider the general quadratic k -th order functional

$$\mathcal{Q}[h] = \int_a^b \sum_{1 \leq i \leq j \leq k} Q_{ij}(t) h^{(i)}(t) h^{(j)}(t) dt,$$

defined on the subspace of $C^\infty([a, b])$ consisting of functions vanishing up to order $k - 1$ at a and b . By repeated integration by parts, \mathcal{Q} can be written as¹

$$(3.1) \quad \mathcal{Q}[h] = \int_a^b P_0(t)h(t)^2 + \dots P_1(t)(\dot{h}(t))^2 + \dots P_k(t)(h^{(k)})^2 dt.$$

The Euler-Lagrange equation of \mathcal{Q} is given by

$$(3.2) \quad P_0 h - \frac{d}{dt}(P_1 \dot{h}) + \dots + (-1)^k \frac{d^k}{dt^k}(P_k h^{(k)}) = 0.$$

We call this equation the *Jacobi equation* of the functional \mathcal{Q} , or, more generally, of a functional whose Hessian is given by \mathcal{Q} . The Jacobi equation is an actual differential equation of order $2k$ if $P_k(t) \neq 0$ for all $t \in [a, b]$; we shall assume the *strict Legendre condition*: $P_k(t)$ is actually positive on the interval $[a, b]$.

3.1. Eswaran identities. Let $\{\sigma_i, i = 1, \dots, k\}$ be a set of linearly independent solutions of (3.2) satisfying

$$(3.3) \quad \sigma_i^{(j)}(a) = 0, \text{ for } i = 1, 2, \dots, k, \text{ and } j = 0, 1, 2, \dots, k - 1,$$

and consider the “sub-Wronskian”

$$(3.4) \quad W[\sigma_1, \sigma_2, \dots, \sigma_k](t) = \det \begin{pmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_k \\ \dot{\sigma}_1 & \dot{\sigma}_2 & \dots & \dot{\sigma}_k \\ \vdots & \vdots & \dots & \vdots \\ \sigma_1^{(k-1)} & \sigma_2^{(k-1)} & \dots & \sigma_k^{(k-1)} \end{pmatrix}.$$

Definition 3.1. A point $t^* \in (a, b]$ is said to be *conjugate* to a if $W[\sigma_1, \sigma_2, \dots, \sigma_k](t^*) = 0$.

We have then the following theorem:

Theorem 3.2. (*Eswaran*) *If there are no conjugate points to a on $(a, b]$, then the functional \mathcal{Q} is positive definite, that is, $\mathcal{Q}(h) \geq 0$ and $\mathcal{Q}(h) = 0$ only when $h \equiv 0$.*

The main idea behind Eswaran’s result is the following identity which allow us, under the disconjugacy hypothesis, to write the quadratic functional \mathcal{Q} as a perfect square plus a total differential:

¹In general, this is not possible in the higher dimensional case. Compare footnote 18 in chapter 5 of [11].

Theorem 3.3. (*Eswaran, following Eastham*) Let $\sigma_1, \dots, \sigma_k$ a linearly independent set of solutions of (4.1) satisfying (3.3) such that the sub-Wronskian satisfies $W[\sigma_1, \sigma_2, \dots, \sigma_k] \neq 0$ in the interval $(a; b)$. Then for any $h \in C^k([a, b])$, we have the identity

$$(3.5) \quad \sum_{l=0}^k P_l(t)(h^{(l)})^2 = P_k \left(\frac{W[h, \sigma_1, \sigma_2, \dots, \sigma_k]}{W[\sigma_1, \sigma_2, \dots, \sigma_k]} \right)^2 + \frac{dR}{dt},$$

where

$$W[h, \sigma_1, \sigma_2, \dots, \sigma_k] = \det \begin{pmatrix} h & \sigma_1 & \sigma_2 & \cdots & \sigma_k \\ \dot{h} & \dot{\sigma}_1 & \dot{\sigma}_2 & \cdots & \dot{\sigma}_k \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h^{(k-1)} & \sigma_1^{(k-1)} & \sigma_2^{(k-1)} & \cdots & \sigma_k^{(k-1)} \\ h^{(k)} & \sigma_1^{(k)} & \sigma_2^{(k)} & \cdots & \sigma_k^{(k)} \end{pmatrix},$$

where R is a rational expression in $\sigma_i, h^{(i)}$ and P_i , such that $R(t^*) = 0$ if $h^{(i)}(t^*) = 0$ for all $i = 0, 1, 2, \dots, k-1$.

Thus, under the hypothesis of theorem 3.3, we can rewrite \mathcal{Q} as

$$\mathcal{Q}(h) = \int_a^b \sum_{l=0}^k P_l(t) h^{(l)} dt = \int_a^b \left(P_k \left(\frac{W[h, \sigma_1, \sigma_2, \dots, \sigma_k]}{W[\sigma_1, \sigma_2, \dots, \sigma_k]} \right)^2 + \frac{dR}{dt} \right) dt,$$

from which theorem 3.2 immediately follows.

These kind of identities were studied by Cimino and Picone [4, 24] in the order one case (second order Euler-Lagrange equation), Leighton and Kreith [17, 15] for second order case (fourth order Euler-Lagrange equation), and in for general order (one-dimensional) variational problems by Eastham [9].

Easwaran [10] uses the identity developed in [9] to attain the minimality conditions in the one dimensional case. Coppel [5] also develops the minimality conditions for extremals of higher order one-dimensional variational problems assisted by the Legendre transform, in order to be in the context of linear Hamiltonian systems. In section 4, we observe that, by being careful about the order of multiplication of certain matrices, Coppel's approach can be extended to arbitrary finite dimensional problems.

3.2. Jacobi curves. Let us note that theorem 3.2 is projective in two senses: first, if we choose a different set of linearly independent solutions, say $\eta_i(t) = \sum a_{ij} \sigma_j(t)$ for some constant invertible $k \times k$ matrix, the conjugacy condition is the same. This means that the conjugacy condition depends only on the subspace (of the space of all solutions)

defined by the vanishing of the first k derivatives, which we call the *vertical* subspace.

More importantly, we do not need to have actual solutions but only their (simultaneous) projective class: if $\phi : [a, b] \rightarrow \mathbb{R}$ is a never-vanishing differentiable function and we substitute each $\sigma_i(t)$ by $\eta_i(t) = \phi(t)\sigma_i(t)$, then $W[\eta_1, \eta_2, \dots, \eta_k](t) = \phi^k(t)W[\sigma_1, \sigma_2, \dots, \sigma_k](t)$ and therefore their zeros coincide.

This motivates us to consider the following moving frame in \mathbb{R}^{2k} : let $\sigma_1(t), \dots, \sigma_k(t), \sigma_{k+1}(t), \dots, \sigma_{2k}(t)$ be linearly independent solutions of the Euler-Lagrange equations (4.1), where the first k solutions vanish up to order $k - 1$ at $t = a$, and

$$\mathcal{C}(t) = \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_k(t) \\ \sigma_{k+1}(t) \\ \vdots \\ \sigma_{2k}(t) \end{pmatrix}.$$

Let now $p(t)$ denote the class of \mathcal{C} under the projection to $\mathbb{R}P^{2k-1}$. We have

Proposition 3.4. *The curve $p(t)$ is fanning as a curve in the divisible Grassmannian.*

Proof. This follows directly from the fact that the determinant of the juxtaposed matrix $(\mathcal{C}(t)|\mathcal{C}'(t)|\dots|\mathcal{C}^{2k-1}(t))$ is the Wronskian of the $2k$ linearly independent solutions of the Jacobi equation. \square

We now consider the frame obtained by the $k - 1$ prolongation of \mathcal{C} ,

$$\mathcal{A}(t) = \begin{pmatrix} \sigma_1(t) & \dot{\sigma}_1(t) & \dots & \sigma_1^{(k-1)}(t) \\ \vdots & \vdots & & \vdots \\ \sigma_k(t) & \dot{\sigma}_k(t) & \dots & \sigma_k^{(k-1)}(t) \\ \sigma_{k+1}(t) & \dot{\sigma}_{k+1}(t) & \dots & \sigma_{k+1}^{(k-1)}(t) \\ \vdots & \vdots & & \vdots \\ \sigma_{2k}(t) & \dot{\sigma}_{2k}(t) & \dots & \sigma_{2k}^{(k-1)}(t) \end{pmatrix},$$

Definition 3.5. The *Jacobi curve* $\ell(t)$ is the space spanned by the columns of $\mathcal{A}(t)$.

It follows from theorem 3.4 that $\ell(t)$ is k -dimensional, that is, ℓ is a curve in the half-Grassmannian $Gr(k, 2k)$. Now if we define the *vertical*

space $\mathcal{V} \subset \mathbb{R}^{2k}$ as the vectors having vanishing first k -coordinates, we have that $\mathcal{V} = \ell(a)$ and theorem 3.2 translates to

Theorem 3.6. *If $\ell(t^*) \cap \mathcal{V} = \{0\}$ for all $t^* \in (a, b]$, then the functional $\mathcal{Q}(t)$ is positive definite.*

In contrast to the first order variational problems, we have that the curve $\ell(t)$ is *not* fanning. Indeed,

Proposition 3.7. *The rank of the curve $\ell(t)$ is one.*

Proof. It follows immediately by computing the rank of the extended matrix $(\mathcal{A}(t) | \mathcal{A}(t))$; the columns $k, k+1, \dots, 2k-1$ are repeats of the columns $2, \dots, k$ but the first k columns and the last column are linearly independent by proposition 3.4. \square

An important feature of the curve $p(t)$ is its *symplectic* behavior. This is better observed after applying the Legendre transform which will be done, in general dimension, in the next section. We shall see then that the canonical flag given by the successive prolongations of $p(t)$ are isotropic-Lagrangian-coisotropic, according to the dimension.

4. THE SYMPLECTIC PROJECTIVE GEOMETRY OF HIGHER ORDER QUADRATIC FUNCTIONALS: THE GENERAL CASE

We now consider a quadratic functional

$$\mathcal{Q} = \int_a^b L(t, \dot{h}(t), \ddot{h}(t), \dots, h^{(k)}(t)) dt$$

where

$$L(t, h, \dot{h}, \dots, h^{(k)}) = \sum_{1 \leq i \leq j \leq k} h^{(i)\top} M_{ij}(t) h^{(j)}$$

and now $h : [a, b] \rightarrow \mathbb{R}^n$ is a vector-valued smooth function, vanishing up to order $k-1$ at the ends of the interval, as in section 2.1. In general, non-commutativity will not let us transform \mathcal{Q} into a functional of the form (3.1). We shall only assume that $M_{jj}(t)$ is symmetric (which is harmless since $v^\top A v$ vanishes if A is antisymmetric) and the *strong Legendre condition*: $M_{kk}(t)$, in addition to being symmetric, *positive definite* for all $t \in [a, b]$. It is easy to see that by integrating by parts, we can make $M_{ij}(t) = 0$ if $j > i+1$ or if $j < i$, and then L can be written as

$$L(t, h, \dot{h}, \dots, h^{(k)}) = \frac{1}{2} \sum_{i=0}^k h^{(i)\top} M_{ii} h^{(i)} + \sum_{i=0}^{k-1} h^{(i)\top} M_{i(i+1)} h^{(i+1)},$$

where $M_{ii} = M_{ii}^T$ and M_{kk} is positive definite. This reduction greatly simplifies the computations. From now on we drop the independent variable t from the notation.

The Euler-Lagrange equation (which we again call Jacobi, equation, if the quadratic functional is the Hessian of a functional along an extremal) is then

$$(4.1) \quad \frac{\partial L}{\partial q_0} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_1} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial q_2} \right) + \dots + (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q_k} \right) = 0.$$

Where the partial derivatives above can be easily obtained from the form of L :

$$\begin{aligned} \frac{\partial L}{\partial q_0} &= M_{00}h + M_{01}\dot{h}, \\ \frac{\partial L}{\partial q_j} &= M_{jj}h^{(j)} + M_{(j-1)j}^T h^{(j-1)} + M_{j(j+1)} h^{(j+1)}, \text{ for } j = 1, \dots, k-1, \\ \frac{\partial L}{\partial q_k} &= M_{kk}h^{(k)} + M_{(k-1)k}^T h^{(k-1)}. \end{aligned}$$

These relations can be written in the following form

$$\begin{pmatrix} \frac{\partial L}{\partial q_0} \\ \vdots \\ \frac{\partial L}{\partial q_{k-1}} \end{pmatrix} = C(t) \begin{pmatrix} h \\ \vdots \\ h^{(k-1)} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial q_k} \end{pmatrix},$$

where C is a curve of $kn \times kn$ matrices that have the M_{ij} blocks arranged as

$$C(t) = \begin{pmatrix} M_{00} & M_{01} & 0 & 0 & \dots & 0 \\ M_{01}^T & M_{11} & M_{12} & 0 & \dots & 0 \\ 0 & M_{12}^T & M_{22} & M_{23} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & M_{(k-3)(k-2)}^T & M_{(k-2)(k-2)} & M_{(k-2)(k-1)}^T \\ 0 & \dots & 0 & 0 & M_{(k-2)(k-1)}^T & \tilde{M}_{(k-1)(k-1)} \end{pmatrix},$$

and $\tilde{M}_{(k-1)(k-1)} = M_{(k-1)(k-1)} - M_{(k-1)k} M_{kk}^{-1} M_{(k-1)k}^T$.

4.1. The Legendre transform and Hamiltonian presentation.

As remarked at the end of section 3.1, a Hamiltonian version of the Jacobi equation, using a higher dimensional extension of the methods of [5], will greatly simplify the computations needed to establish the Jacobi curve and its relationship with minimality.

Consider then the *Legendre transform* $\text{Leg} : \mathbb{R}^{2kn} \rightarrow \mathbb{R}^{2kn}$ associated to L . The Legendre transform applied to the $(2k-1)$ -jet of a curve h

is given by:

$$\text{Leg}(h, \dot{h}, \dots, h^{(k-1)}, h^{(k)}, \dots, h^{(2k-1)}) = (y, z)$$

with

$$y = \begin{pmatrix} h \\ \dot{h} \\ \ddot{h} \\ \vdots \\ h^{(k-1)} \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_k \end{pmatrix}$$

where

$$z_i = \sum_{j=i}^k (-1)^{j-i} \left(\frac{d}{dt} \right)^{j-i} \left(\frac{\partial L}{\partial q_j} \right).$$

Computation shows that the Legendre transform will take solutions of (4.1) into solutions of a Hamiltonian system in the variables (y, z) given by

$$(4.2) \quad \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A(t) & B(t) \\ C(t) & -A(t)^T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

where $C(t)$ is the same as above, $A(t)$ is given by

$$A(t) = \begin{pmatrix} 0 & \text{Id} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \text{Id} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \text{Id} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \text{Id} \\ 0 & 0 & 0 & \cdots & 0 & -M_{kk}^{-1} M_{(k-1)k}^T \end{pmatrix},$$

and $B(t)$ is given by

$$B(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & (M_{kk})^{-1} \end{pmatrix},$$

where the Id and 0 blocks have size $n \times n$ in both matrices above.

Note that $B(t)^T = B(t)$ e $C(t)^T = C(t)$, and then the matrix with the A , B and C blocks in the expression (4.2) will be in the Lie algebra of the symplectic group (with respect to the canonical symplectic form on \mathbb{R}^{2n} making the $(y, 0)$ and $(0, z)$ Lagrangian subspaces). Thus, equation (4.2) is indeed a Hamiltonian system.

Let us define another transform on the k -jets of functions h , which we call the *zeroing transform* since it maps the functional \mathcal{Q} to a functional that apparently involves no derivatives.

Under the zeroing transform, the k -jet of h is mapped to (\hat{y}, \hat{z}) , where

$$(4.3) \quad \hat{y} = \begin{pmatrix} h \\ \dot{h} \\ \vdots \\ h^{(k-2)} \\ h^{(k-1)} \end{pmatrix} \quad \text{and} \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ M_{kk}h^{(k)} + M_{(k-1)k}^T h^{(k-1)} \end{pmatrix},$$

There are two important properties of the zeroing transform: first, even though the pair (\hat{y}, \hat{z}) usually does not satisfies equation 4.2 the equation $\dot{\hat{y}} = A(t)\hat{y} + B(t)\hat{z}(t)$ is an *identity* implied by the form of the matrices A and B . This identity is needed in the generalized Picone identity 4.6.

The zeroing transform also allows the writing of the functional \mathcal{Q} in a simpler way. Again take a function $h \in C^k([a, b], \mathbb{R}^n)$ and the pair (\hat{y}, \hat{z}) defined by (4.3). Then we will have

$$\begin{aligned} \hat{z}^T B \hat{z} &= (\hat{z}_1^T \quad \cdots \quad \hat{z}_k^T) \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & (M_{kk})^{-1} \end{pmatrix} \begin{pmatrix} \hat{z}_1 \\ \vdots \\ \hat{z}_k \end{pmatrix} = \\ &= \hat{z}_k^T M_{kk}^{-1} \hat{z}_k = \left(h^{(k)T} M_{kk} + h^{(k-1)T} M_{(k-1)k} \right) M_{kk}^{-1} \left(M_{kk} h^{(k)} + M_{(k-1)k}^T h^{(k-1)} \right) = \\ &= h^{(k)T} M_{kk} h^{(k)} + 2h^{(k-1)T} M_{(k-1)k} h^{(k)} + h^{(k-1)T} M_{(k-1)k} M_{kk}^{-1} M_{(k-1)k}^T h^{(k-1)}, \end{aligned}$$

and

$$\begin{aligned}
\hat{y}^T C \hat{y} &= \hat{y}^T \left(\begin{pmatrix} \frac{\partial L}{\partial q_0} \\ \vdots \\ \frac{\partial L}{\partial q_{k-1}} \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial q_k} \end{pmatrix} \right) = \\
&= \hat{y}^T \begin{pmatrix} \frac{\partial L}{\partial q_0} \\ \vdots \\ \frac{\partial L}{\partial q_{k-1}} \end{pmatrix} - \hat{y}^T \begin{pmatrix} 0 \\ \vdots \\ M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial q_k} \end{pmatrix} = \\
&= \sum_{i=0}^{k-1} h^{(i)T} M_{ii} h^{(i)} + 2 \sum_{i=0}^{k-2} h^{(i)T} M_{i(i+1)} + h^{(k-1)T} M_{(k-1)k} h^{(k)} + \\
&\quad - h^{(k-1)T} M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial q_k} = \\
&= \sum_{i=0}^{k-1} h^{(i)T} M_{ii} h^{(i)} + 2 \sum_{i=0}^{k-2} h^{(i)T} M_{i(i+1)} h^{(i+1)} + h^{(k-1)T} M_{(k-1)k} h^{(k)} \\
&\quad - \left(h^{(k-1)T} M_{(k-1)k} h^{(k)} + h^{(k-1)T} M_{(k-1)k} M_{kk}^{-1} M_{(k-1)k}^T h^{(k-1)} \right) = \\
&= \sum_{i=0}^{k-1} h^{(i)T} M_{ii} h^{(i)} + 2 \sum_{i=0}^{k-2} h^{(i)T} M_{i(i+1)} h^{(i+1)} - h^{(k-1)T} M_{(k-1)k} M_{kk}^{-1} M_{(k-1)k}^T h^{(k-1)}.
\end{aligned}$$

Adding $\hat{z}^T B \hat{z}$ and $\hat{y}^T C \hat{y}$ leads to

$$\hat{z}^T B \hat{z} + \hat{y}^T C \hat{y} = \sum_{i=0}^k h^{(i)T} M_{ii} h^{(i)} + 2 \sum_{i=0}^{k-1} h^{(i)T} M_{i(i+1)} h^{(i+1)} = 2L.$$

Then, the Legendre transform takes solutions of (4.1) to solutions of (4.2), and the zeroing transform will give the equality of the quadratic functionals (up to a factor 2):

$$\tilde{\mathcal{Q}} = \int_a^b \hat{z}^T B \hat{z} + \hat{y}^T C \hat{y} dt = 2\mathcal{Q} = 2 \int_a^b L(t, \dot{h}(t), \ddot{h}(t), \dots, h^{(k)}(t)) dt.$$

Let h_1, \dots, h_{kn} be a linearly independent set of solutions of (4.1) such that all the derivatives up to the $k-1$ order vanish at $t = a$. Consider the following sub-Wronskian

$$W[h_1, \dots, h_{kn}](t) = \det \begin{pmatrix} h_1^T(t) & \dot{h}_1^T(t) & \dots & h_1^{(k-1)T}(t) \\ h_2^T(t) & \dot{h}_2^T(t) & \dots & h_2^{(k-1)T}(t) \\ \vdots & \vdots & \vdots & \vdots \\ h_{kn}^T(t) & \dot{h}_{kn}^T(t) & \dots & h_{kn}^{(k-1)T}(t) \end{pmatrix}$$

Definition 4.1. A point $t^* \in (a, b]$ is said to be *conjugate* to a if $W[h_1, \dots, h_{kn}](t^*) = 0$.

Then the following is immediate:

Lemma 4.2. *A point t^* is conjugate to a if, and only if, exists a non-trivial solution h of (4.1) such that $h^{(i)}(a) = h^{(i)}(t^*) = 0$, for $i = 0, 1, \dots, k - 1$.*

In the same way as 1 dimensional case we will have

Theorem 4.3. *If there are no conjugate points to a on $(a, b]$, then the functional \mathcal{Q} is positive definite, that is, $\mathcal{Q}(h) \geq 0$ and $\mathcal{Q}(h) = 0$ only when $h \equiv 0$.*

The proof of this theorem will involve a generalized Picone identity of the Hamiltonian system (4.2). Let us fix a set $\{h_1, \dots, h_{kn}\}$ of linearly independent solutions of (4.1) such that all the derivatives up to the $k - 1$ order vanish at $t = a$, and consider the image of the Legendre transform of the $(2k - 1)$ -jet of each h_j ,

$$\text{Leg} \left(h_j, \dot{h}_j, \dots, h_j^{(2k-1)} \right) = \begin{pmatrix} \mu_j \\ \zeta_j \end{pmatrix},$$

we now put then together by constructing the $2kn \times kn$ matrix where the columns are the image of the Legendre transform for each h_j :

$$\begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} \mu_1 & \cdots & \mu_j \\ \zeta_1 & \cdots & \zeta_j \end{pmatrix},$$

which defines

$$Y(t) = (\mu_1 \ \cdots \ \mu_j) \quad \text{and} \quad Z(t) = (\zeta_1 \ \cdots \ \zeta_j).$$

We have the following lemma, where the proof follows by the fact that each image of the Legendre transform of h_j satisfy 4.2 and using the initial condition at $t = a$:

Lemma 4.4. *In the conditions above we have*

$$Y(t)^T Z(t) - Z(t)^T Y(t) = 0, \forall t.$$

Corollary 4.5. *In the conditions above, and supposing that $Y(t)$ is invertible for all t , we have that $Z(t)Y(t)^{-1}$ is symmetric for all t .*

We can now state the generalized Picone identity:

Theorem 4.6 (Generalized Picone Identity). *Let $(Y; Z)$ satisfy 4.4 above and suppose that $Y(t)$ is invertible for all $t \in (a; b]$. Also consider*

the pair $(y; z)$ satisfying the relation $\dot{y} = A(t)y + B(t)z$. In these conditions we have

$$(4.4) \quad \frac{d}{dt}(y^T ZY^{-1}y) = z^T Bz + y^T Cy - (z - ZY^{-1}y)^T B(z - ZY^{-1}y).$$

Proof. Calculating the derivative above, using $\dot{y} = A(t)y + B(t)z$ and

$$\dot{Y} = AY + BZ, \quad \dot{Z} = CY - A^T Z,$$

we have

$$\begin{aligned} \frac{d}{dt}(y^T ZY^{-1}y) &= \dot{y}^T ZY^{-1}y + y^T \dot{Z}Y^{-1}y + y^T Z(-Y^{-1}\dot{Y}Y^{-1})y + y^T ZY^{-1}\dot{y} = \\ &= (Ay + Bz)^T ZY^{-1}y + y^T (CY - A^T Z)Y^{-1}y - y^T Z(Y^{-1}(AY + BZ)Y^{-1})y + \\ &+ y^T ZY^{-1}(Ay + Bz) = \\ &= (\cancel{y^T A} + z^T B)ZY^{-1}y + y^T (C - \cancel{A^T ZY^{-1}})y - y^T Z(\cancel{Y^{-1}A} + Y^{-1}BZY^{-1})y + \\ &+ y^T ZY^{-1}(\cancel{Ay}) + Bz) = \\ &= z^T BZY^{-1}y + y^T Cy - y^T ZY^{-1}BZY^{-1}y + y^T ZY^{-1}Bz + z^T Bz - z^T Bz = \\ &= z^T B + y^T Cy + \underbrace{(z^T BZY^{-1}y - y^T ZY^{-1}BZY^{-1}y + y^T ZY^{-1}Bz - z^T Bz)}_{\Delta}. \end{aligned}$$

Expanding the last term in (4.4) and using that ZY^{-1} is symmetric we have

$$\begin{aligned} - (z - ZY^{-1}y)^T B(z - ZY^{-1}y) &= (y^T (ZY^{-1})^T - z^T)B(z - ZY^{-1}y) = \\ &= (y^T ZY^{-1} - z^T)B(z - ZY^{-1}y) = y^T ZY^{-1}Bz - y^T ZY^{-1}BZY^{-1}y - z^T Bz + \\ &+ z^T BZY^{-1}y = \Delta, \end{aligned}$$

and then, the identity follows. \square

With the hypothesis above, let $h \in C^k([a, b], \mathbb{R}^n)$ satisfying $h^{(i)}(a) = h^{(i)}(b) = 0$, $0 \leq i \leq k-1$, and consider the image (\hat{y}, \hat{z}) of the zeroing transform of the k -jet of h (as in (4.3)). We have

$$\begin{aligned} 2\mathcal{Q}[h] &= \tilde{\mathcal{Q}}[\hat{y}, \hat{z}] = \int_a^b \hat{z}^T B\hat{z} + \hat{y}^T C\hat{y} dt = \\ &= \int_a^b \frac{d}{dt}(\hat{y}^T ZY^{-1}\hat{y}) + (\hat{z} - ZY^{-1}\hat{y})^T B(\hat{z} - ZY^{-1}\hat{y}) dt = \\ &= \int_a^b (\hat{z} - ZY^{-1}\hat{y})^T B(\hat{z} - ZY^{-1}\hat{y}) dt \geq 0 \end{aligned}$$

and

$$\begin{aligned}
 2\mathcal{Q} = \tilde{\mathcal{Q}} = 0 &\Leftrightarrow \int_a^b (\hat{z} - ZY^{-1}\hat{y})^T B(\hat{z} - ZY^{-1}\hat{y}) dt = 0 \Leftrightarrow \\
 &\Leftrightarrow B(\hat{z} - ZY^{-1}\hat{y}) = 0 \Leftrightarrow \dot{\hat{y}} - A\hat{y} - BZY^{-1}\hat{y} = 0 \Leftrightarrow \\
 &\Leftrightarrow \dot{\hat{y}} = (A + BZY^{-1})\hat{y} \Leftrightarrow \hat{y} \equiv 0.
 \end{aligned}$$

Then theorem 4.3 follows, since $\hat{y} \equiv 0$ implies that $h \equiv 0$.

4.2. Jacobi curves. Let us projectivize as in the 1-dimensional case. Define the $2kn \times n$ frame

$$(4.5) \quad \mathcal{A}(t) = \begin{pmatrix} h_1^T \\ \vdots \\ h_{kn}^T \\ h_{kn+1}^T \\ \vdots \\ h_{2kn}^T \end{pmatrix},$$

where $\{h_1, \dots, h_{2kn}\}$ is a fundamental set of solutions of (4.1) such that for $i = 1, \dots, kn$, each h_i has all derivatives vanishing up to order $k-1$ at $t = a$.

Again considering the space $p(t)$ spanned by the columns of \mathcal{A} at each t we will have

Theorem 4.7. *The curve $p(t)$ is a fanning curve in the Grassmannian $Gr(n, 2kn)$*

Proof. Considering the prolongation to $(2k-1)$ -jet of p written in terms of the frame \mathcal{A} , the rank of this prolongation is maximal, i.e., for each t the $2kn \times 2kn$ matrix below is non-degenerate:

$$\left(\mathcal{A}(t) \middle| \dot{\mathcal{A}}(t) \middle| \dots \middle| \mathcal{A}^{(2k-1)}(t) \right).$$

The non-degeneracy comes from the fact that the determinant of the matrix above is the Wronskian of a set of fundamental solutions of (4.1) (which is non-zero for all t), and then the assertion follows. \square

Definition 4.8. The *Jacobi curve* $\ell : [a, b] \rightarrow Gr(kn, 2kn)$ is the $(k-1)$ -jet prolongation of p , that is, the curve of subspaces spanned by the columns of the matrix

$$(4.6) \quad \left(\mathcal{A}(t) \middle| \dot{\mathcal{A}}(t) \middle| \dots \middle| \mathcal{A}^{(k-1)}(t) \right).$$

Now if we define the *vertical space* $\mathcal{V}_{kn}^{2kn} \subset \mathbb{R}^{2k}$ as the vectors having vanishing first kn -coordinates, we have that $\mathcal{V}_{kn}^{2kn} = \ell(a)$ and theorem 4.3 translates to

Theorem 4.9. *If $\ell(t^*) \cap \mathcal{V}_{kn}^{2kn} = \{0\}$ for all $t^* \in (a, b]$, then the functional $\mathcal{Q}(t)$ is positive definite.*

Therefore the lack of self-intersections of the Jacobi curve implies the positivity of the quadratic functional \mathcal{Q} .

In contrast with first order variational problems, this Jacobi curve is highly degenerate. We have:

Theorem 4.10. *The rank of the Jacobi curve $\ell(t)$ is n .*

Proof. Proposition 2.2 can be applied to the frame of (4.6) of $\ell(t)$. The rank of ℓ is then given by

$$\text{rank} \left(\mathcal{A}(t) \middle| \cdots \middle| \mathcal{A}^{(k-1)}(t) \middle| \dot{\mathcal{A}}(t) \middle| \cdots \middle| \mathcal{A}^{(k)}(t) \right) - kn,$$

which, by theorem 4.7, is $n(k+1) - kn = n$. \square

Observe that the rank of a fanning curve in the half-Grassmannian $\text{Gr}(kn, 2kn)$ is kn ; thus the Jacobi curve of higher order variational problems ($k > 1$) is *never* fanning.

4.3. Isotropic-Lagrangian-Coisotropic Flag. We now study the symplectic properties of the fanning curve $p : [a, b] \rightarrow \text{Gr}(n, 2kn)$ and its prolongations. Endow \mathbb{R}^{2kn} with the canonical symplectic form ω_{can} induced by the decomposition $\mathbb{R}^{2kn} \cong \mathbb{R}^{kn} \oplus \mathbb{R}^{kn}$ in its first and last kn coordinates. Then we have

Theorem 4.11. *Considering the symplectic space $(\mathbb{R}^{2kn}, \omega_{\text{can}})$, the curve $p : [a, b] \rightarrow \text{Gr}(n, 2kn)$ defined in the last section satisfies*

- $j^i p : [a, b] \rightarrow \text{Gr}((i+1)n, 2kn)$ is a curve of isotropic subspaces for $i = 0, \dots, k-2$,
- $\ell = j^{k-1} p : [a, b] \rightarrow \text{Gr}(kn, 2kn)$ is a curve of Lagrangian subspaces,
- $j^i p : [a, b] \rightarrow \text{Gr}((i+1)n, 2kn)$ is a curve of coisotropic subspaces for $i = k, \dots, 2k-1$.

Proof. The proof relies on the invariance of the isotropic-Lagrangian-coisotropic concepts under certain “tiltings” of the Legendre transform. Recall from the beginning of this section that the Legendre transform has the form

$$\text{Leg}(h, \dots, h^{(2k-1)}) = \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} h \\ \vdots \\ h^{(2k-1)} \end{pmatrix}.$$

Where we now have to study in more detail the matrices B_1 and B_2 . In the expression above the blocks Id , 0 , B_1 and B_2 are $kn \times kn$ matrices. The block B_1 is an upper triangular matrix and the block B_2 has blocks M_{kk} or $-M_{kk}$ in the anti-diagonal and zeros below the anti-diagonal, that is, B_2 is of the form

$$B_2 = \begin{pmatrix} * & * & \cdots & * & (-1)^{k-1}M_{kk} \\ * & * & \cdots & (-1)^{k-2}M_{kk} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & -M_{kk} & 0 & \cdots & 0 \\ M_{kk} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From this we get that each coordinate z_i will depend (linearly) only of $h^{(i-1)}, h^{(i)}, \dots, h^{(2k-i)}$, for $i = 1, \dots, k$.

Now for each h_i given in (4.5) consider the image (μ_i, ζ_i) of the Legendre transform of the $(2k-1)$ -jet prolongation of h_i

$$\text{Leg}(h_i, \dots, h_i^{(2k-1)}) = \begin{pmatrix} \mu_i \\ \zeta_i \end{pmatrix},$$

and construct the following matrix

$$\begin{pmatrix} \mu_1 & \cdots & \mu_{2kn} \\ \zeta_1 & \cdots & \zeta_{2kn} \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} h_1 & \cdots & h_{2kn} \\ \vdots & \vdots & \vdots \\ h_1^{(2k-1)} & \cdots & h_{2kn}^{(2k-1)} \end{pmatrix}.$$

Calculating the transpose in the relation above will lead us to

$$\begin{aligned} (4.7) \quad \begin{pmatrix} \mu_1^T & \zeta_1^T \\ \vdots & \vdots \\ \mu_{2kn}^T & \zeta_{2kn}^T \end{pmatrix} &= \left(\mathcal{A}(t) \middle| \dot{\mathcal{A}}(t) \middle| \cdots \middle| \mathcal{A}^{(2k-1)}(t) \right) \begin{pmatrix} \text{Id} & B_1^T \\ 0 & B_2^T \end{pmatrix} \\ &= \left(\mathcal{A}(t) \middle| \dot{\mathcal{A}}(t) \middle| \cdots \middle| \mathcal{A}^{(k-1)}(t) \middle| \mathcal{C}^1(t) \middle| \cdots \middle| \mathcal{C}^k(t) \right) \end{aligned}$$

where the $2kn \times n$ blocks \mathcal{C}^i are linear combinations of the columns of $\mathcal{A}^{(i-1)}, \dots, \mathcal{A}^{(2k-i)}$, for $i = 1, \dots, k$, that can be written as

$$\mathcal{C}^i = \mathcal{A}^{(i-1)}Q_{i-1}^i + \dots + \mathcal{A}^{(2k-i)}Q_{2k-i}^i,$$

where the Q_j^i are $n \times n$ matrices and, most importantly, $Q_{2k-i}^i = \pm M_{kk}$.

Now, the matrix on the left side of the equality (4.7) will be in the Lie group of symplectic matrices if we suppose the initial condition

$$\begin{pmatrix} \mu_1^T(a) & \zeta_1^T(a) \\ \vdots & \vdots \\ \mu_{2kn}^T(a) & \zeta_{2kn}^T(a) \end{pmatrix} = \begin{pmatrix} 0_{kn} & \text{Id}_{kn} \\ -\text{Id}_{kn} & 0_{kn} \end{pmatrix}.$$

Then we will have that the matrices in (4.7) are symplectic for each t and the initial condition makes the first columns vanish up to order $k - 1$ at $t = a$, as required.

Denoting by J the matrix of the canonical symplectic form ω_{can} in \mathbb{R}^{2kn} , we will have that

$$(4.8) \quad \left(\mathcal{A}(t) \middle| \cdots \middle| \mathcal{C}^k(t) \right)^T J \left(\mathcal{A}(t) \middle| \cdots \middle| \mathcal{C}^k(t) \right) = J = \begin{pmatrix} 0_{kn} & \text{Id}_{kn} \\ -\text{Id}_{kn} & 0_{kn} \end{pmatrix}.$$

which in turn implies

$$(4.9) \quad \mathcal{A}^{(i-1)T} J \mathcal{A}^{(j-1)} = 0,$$

for $i, j = 1, \dots, k$, and

$$\mathcal{C}^{iT} J \mathcal{A}^{(j-1)} = 0$$

for $2 \leq i \leq k$ and $1 \leq j \leq i - 1$. Developing further the second expression using that \mathcal{C}^i can be written as

$$\mathcal{C}^i = \mathcal{A}^{(i-1)} Q_{i-1}^i + \dots + \mathcal{A}^{(2k-i)} Q_{2k-i}^i,$$

with $Q_{2k-i}^i = \pm M_{kk}$ (that is non-degenerate), we will have that

$$(4.10) \quad \mathcal{A}^{(i)T} J \mathcal{A}^{(j)} = 0,$$

for $k \leq i \leq 2k - 2$ and $0 \leq j \leq 2k - i - 2$.

Now (4.9) and (4.10) implies all items of theorem 4.11 simultaneously.

□

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